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## TECHNICAL NOTE

# Coefficient of consolidation by plotting velocity against displacement

J. D. MCKINLEY and V. SIVAKUMAR

A method is presented for estimating the initial compression, the final compression and the coefficient of consolidation from an observed, experimental consolidation response, using a plot of velocity versus displacement and the conventional Taylor plot of compression versus the square root of time. Goodness of fit measures indicate that the method produces good agreement between fitted and measured displacement values, at least up until the point where the impact of secondary compression on the overall displacement response becomes significant.

**KEYWORDS:** clays; consolidation; laboratory tests

La présente communication illustre une méthode d'estimation de la compression initiale, de la compression finale, et du coefficient de consolidation, à partir d'une réaction de consolidation expérimentale observée, en faisant usage d'un tracé de la vitesse en fonction du déplacement, ainsi que du tracé traditionnel compression /racine carrée du temps de Taylor. Les mesures de la qualité de la tenue indiquent que cette méthode permet d'obtenir un bon accord entre valeurs de déplacement installé et mesuré, au moins jusqu'au point où l'impact de la compression secondaire sur la réponse générale au déplacement devient significatif.

## INTRODUCTION

The determination of the coefficient of consolidation  $c_v$  from the results of a one-dimensional laboratory consolidometer test is an area of considerable practical interest to geotechnical engineers. Methods by Taylor and Casagrande are widely used to estimate  $c_v$  (Taylor, 1948). Both methods derive from approximate forms of the solution for Terzaghi's one-dimensional primary consolidation problem with an initially uniform excess pore pressure distribution. Both techniques rely on only part of the actual data, where Taylor's method weights the early part of the consolidation behaviour and Casagrande's method weights the later part of the consolidation behaviour.

The determination of model parameters fitting an experimental response requires both an assessment of the range over which the model being fitted is a reasonable description of the observed response and an estimation of the values of the parameters in the model. In a least squares regression fitting approach the analyst must specify which portion of the actual data is to be fitted (see, for example, the work on least squares estimation of consolidation parameters in McKinley (1993) Robinson & Allam (1998) and Chan (2003)). A major issue for consolidometer analysis is secondary compression. For example, distinguishing between primary consolidation and secondary compression using pore pressure measurements, Robinson (2003) found that for inorganic soils secondary compression becomes significant for a degree of consolidation greater than about 92% consolidation if the load-increment ratio is one or greater. For conventional tests on inorganic soils with load-increment ratios of one, however, it is generally accepted that secondary compression has little impact until relatively late in the test.

This technical note presents a method for estimating the corrected zero point displacement gauge reading,  $\delta_s$ , the displacement gauge reading at 100% of primary consolida-

tion,  $\delta_{100}$ , and the coefficient of vertical consolidation,  $c_v$ , from a laboratory consolidation response, using plots of velocity against displacement and the conventional Taylor plot. Comparison with the Taylor method is made. Divergence of the observed response from the Terzaghi primary consolidation model is also considered.

## THEORETICAL DEVELOPMENT OF THE VELOCITY DISPLACEMENT METHOD

The following theoretical development is based on an analysis originally undertaken by Doran (2003, personal communication). Fox (1948) proposed a simple two-section approximate solution for Terzaghi's one-dimensional primary consolidation problem with an initially uniform excess pore pressure distribution. In Fox's solution the relationship between displacement gauge reading  $\delta$  and time  $t$  is

$$\frac{\delta - \delta_s}{\delta_{100} - \delta_s} = 2\sqrt{\frac{c_v t}{\pi d^2}} \text{ for } \frac{c_v t}{d^2} < 0.213 \quad (1)$$

and

$$\frac{\delta - \delta_s}{\delta_{100} - \delta_s} = 1 - \frac{8}{\pi^2} \exp\left(-\frac{\pi^2 c_v t}{4d^2}\right) \text{ for } \frac{c_v t}{d^2} \geq 0.213 \quad (2)$$

where  $d$  is the length of the drainage path. For ease of notation, it is usual to express both displacement and time in non-dimensional form, where the degree of consolidation  $U$  and the time-factor  $T_v$  are defined as

$$U = \frac{\delta - \delta_s}{\delta_{100} - \delta_s} \quad (3)$$

and

$$T_v = \frac{c_v t}{d^2} \quad (4)$$

respectively. In Fox (1948) the transition point between equations (1) and (2) is reported as occurring at  $T_v = 0.2$ , but Doran & McKinley (2006) report an improved estimate of 0.213. The transition point from equation (1) to equation (2) occurs at  $U \approx 0.521$ . Fox's approximation produces a

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maximum relative error in the degree of consolidation of 0.2% (Doran & McKinley, 2006). This occurs at the transition point.

Differentiating equation (1) gives

$$\frac{d\delta}{dt} = (\delta_{100} - \delta_s) \sqrt{\frac{c_v}{\pi d^2 t}} \quad (5)$$

and when equation (1) is substituted into equation (5)

$$\delta - \delta_s = \frac{2c_v}{\pi d^2} (\delta_{100} - \delta_s)^2 \left( \frac{d\delta}{dt} \right)^{-1} \text{ for } U < 0.521 \quad (6)$$

Equation (6) indicates that a plot of slowness, that is one over velocity, plotted against displacement should be a straight line for  $U \leq 50\%$  approximately. The intercept on the slowness axis occurs at  $\delta = \delta_s$ . However,  $\delta_s$  is usually estimated from a Taylor plot (Taylor, 1948).

Differentiating equation (2) gives

$$\frac{d\delta}{dt} = 2(\delta_{100} - \delta_s) \frac{c_v}{d^2} \exp\left(-\frac{\pi^2 c_v t}{4d^2}\right) \quad (7)$$

$$\delta - \delta_{100} = -\frac{4d^2}{\pi^2 c_v} \cdot \frac{d\delta}{dt} \text{ for } U \geq 0.521 \quad (8)$$

when equation (1) is substituted into equation (7). Equation (8) indicates that plot of velocity against displacement should be a straight line for  $U \geq 50\%$  approximately. The intercept on the velocity axis occurs at  $\delta = \delta_{100}$ .

#### PRACTICAL IMPLEMENTATION

The above theoretical development shows that  $\delta_s$  can be obtained from a slowness versus displacement plot,  $\delta_{100}$  from a plot of velocity versus displacement plot, and four separate estimates of  $c_v/d^2$  can be obtained from these two plots. For comparison with the velocity displacement method presented here the authors have first adopted a semi-automated approach to the conventional Taylor method, based on the approach reported by McKinley (1993). This semi-automated Taylor method involves a visual estimate of the minimum and maximum values of  $\sqrt{t}$  in the linear section, least squares linear regression on the linear section, and estimation of  $t_{90}$  by intersection assuming that the displacement is piecewise linear in the square root of time. This was implemented as a VBA macro in Microsoft Excel. All stages are automated except that of estimating the start and finish of the linear section, where engineering judgement must be exercised.

The conventional Taylor's method does not directly lead to an estimate of  $\delta_{100}$ . It does, however, involve identifying  $\delta_{90}$ , corresponding to 90% primary consolidation, and  $\delta_s$ , so  $\delta_{100}$  can be calculated. In addition, a separate estimate of  $c_v/d^2$  is calculated from the gradient of the linear section using equation (1). This method gives an internally consistent estimate of both  $\delta_s$  and  $\delta_{100}$ , and two separate estimates of  $c_v/d^2$ . A similar approach can be taken for the Casagrande method (McKinley, 1993), but is not attempted.

Feng & Lee (2001) also proposed a method in which the linear section on the Taylor plot is used to estimate the coefficient of consolidation. However, they estimate the point where the observed response deviates from the straight line, whereas the method above relies on the gradient of the straight line. It is, however, dependent on the reliability of the estimate of  $\delta_{90}$ .

The practical application of the velocity displacement method developed by the authors is relatively straightforward, and has also been implemented as a semi-automated

approach in Microsoft Excel. In outline, the velocity displacement method is

- calculate, using centred finite differences, the velocity and displacement from the experimental readings
- plot velocity against displacement, and by eye identify the linear section of the plot corresponding approximately to the last 50% of the displacement
- use least squares linear regression on this linear section and obtain  $\delta_{100}$
- plot one over velocity against displacement, and by eye identify the linear section of the plot corresponding approximately to the first 50% of the displacement
- use least squares linear regression on this linear section and obtain  $\delta_s$
- from  $\delta_s$  and  $\delta_{100}$  calculate  $\delta_{50}$ , corresponding to 50% primary consolidation, and estimate  $t_{50}$  by linear interpolation
- from  $t_{50}$  estimate  $c_v/d^2$  using equation (4) and
- from the gradient of the linear section in the velocity against displacement plot, estimate of  $c_v/d^2$  using equation (8).

All stages are automated except that of identifying the two linear sections. Intersection to find the time for any degree of consolidation on the primary consolidation curve could be done, but  $t_{50}$  is used because it is well away from the area where secondary consolidation is likely to become significant.

Applying both the Taylor method, extended to use the initial gradient, and the velocity displacement method developed here, produces two estimates of  $\delta_s$ , two estimates of  $\delta_{100}$  and four estimates of  $c_v/d^2$ . The extent to which these estimates are consistent will indicate the extent to which the underlying model, which here is Fox's solution, is a good description of the observed response during primary consolidation.

#### APPLICATION TO LABORATORY TEST DATA FOR A RECONSTITUTED CLAY

The practical use of the methods above is illustrated using the results of a conventional laboratory consolidometer test on a reconstituted sample of Belfast Upper Boulder Clay from the Pollock Dock, Belfast. This is designated as sample Doran BUBC 2212. The load-increment ratio was one, so secondary compression effects should not be significant during the earlier stages of consolidation (Robinson, 2003). The displacement gauge readings were taken using an automatic data logging system over a period of 24 h.

Figures 1(a) and 1(b) show the conventional Taylor and Casagrande plots, respectively, for sample Doran BUBC 2212. The response is reasonably typical of a conventional consolidometer test on an inorganic clay with testing carried slightly beyond the end of primary consolidation. It shows a substantial section roughly linear in  $\sqrt{t}$ , with a slight delay in the development of movement, and approximately half a logarithmic cycle of secondary compression at the end of the test. The semi-automated Taylor method described above gives  $\delta_s \approx 4.6203$  mm and  $\delta_{100} \approx 3.8153$  mm, so  $\delta_{50} \approx 4.218$  mm.

Figure 1(c) shows the plot of velocity against displacement gauge reading, where the horizontal axis has been ordered so that the first test reading is on the left and the last reading is on the right. The curve is approximately hyperbolic, with an apparently linear section from about 4.3 mm onwards. This is consistent with the analysis above, where the plot would be linear for  $U$  greater than 50%. Extrapolation to zero velocity gives  $\delta_{100} \approx 3.8204$  mm, which is close to the value obtained from Taylor's method.

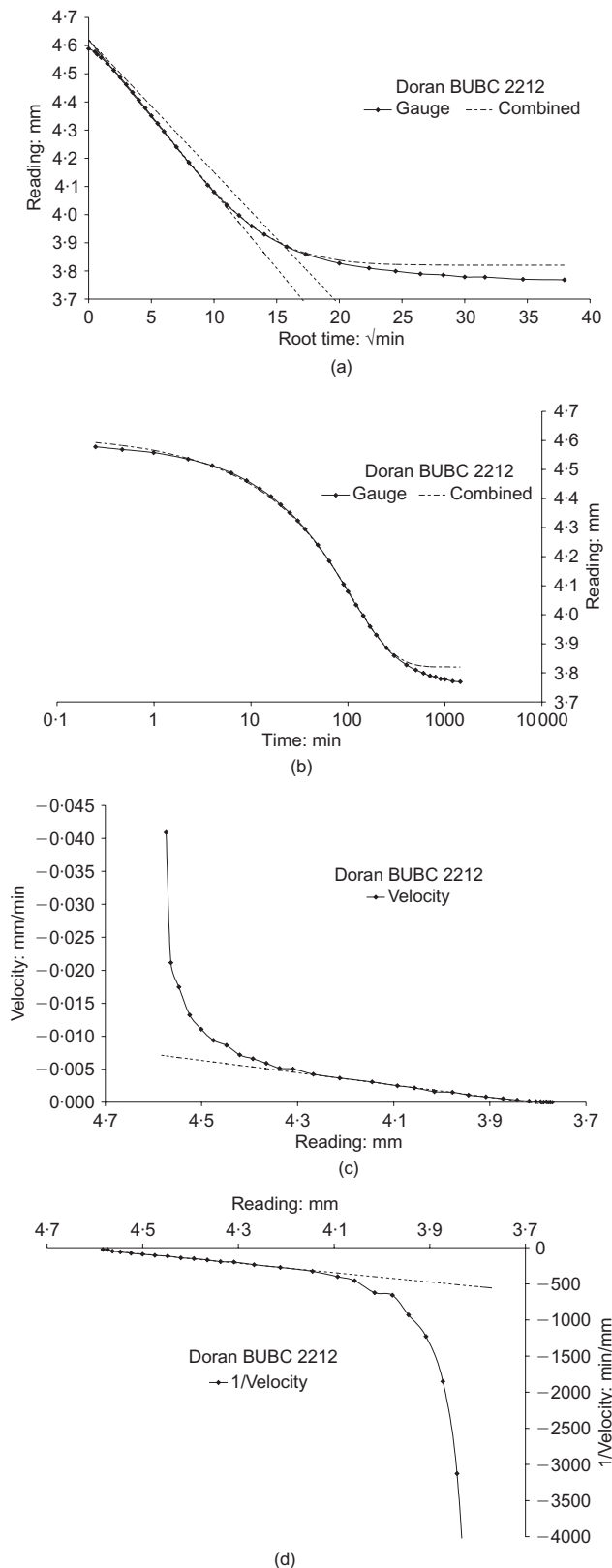


Fig. 1. Taylor, Casagrande, velocity and inverse velocity plots for sample BUBC 2212: (a) Taylor plot; (b) Casagrande plot; (c) velocity plot; (d) inverse velocity plot

Figure 1(d) shows the plot of inverse velocity versus displacement gauge reading; again, time increases from left to right. The curve is approximately hyperbolic, with an apparently linear section up to about 4.1 mm. This is consistent with the analysis above, where the plot would be linear for  $U$  less than 50%. Extrapolation on to the displace-

ment gauge reading axis gives  $\delta_s \approx 4.6352$  mm, which again is to the value obtained from Taylor's method.

The conventional Taylor's method gives  $c_v/d^2 \approx 0.003570/\text{min}$ , the modified Taylor method using the gradient of the linear section gives  $c_v/d^2 \approx 0.003542/\text{min}$ , the method based on identifying  $\delta_{100}$  and  $t_{50}$  using a plot of velocity against displacement gives  $c_v/d^2 \approx 0.003754/\text{min}$ , and the method based on the gradient of the linear section in the plot of velocity versus displacement gives  $c_v/d^2 \approx 0.003776/\text{min}$ . These four values are in close agreement, and averaging them gives  $c_v/d^2 \approx 0.00366/\text{min}$ .

#### CONSIDERATION OF GOODNESS OF FIT

Examination of the residuals, that is, the difference between the observed values and those predicted by the fitted response, is an important tool for the consideration of whether the model used is a reasonable description of the observed behaviour, and whether the underlying assumptions in the model fitting procedure are likely to be appropriate (Draper & Smith, 1998). To this end, three fitted responses based on Fox's approximate solution were generated for the sample Doran BUBC 2212: the first, using the  $\delta_s$ ,  $\delta_{100}$ , and  $c_v/d^2$  values from the conventional Taylor method; the second, using the  $\delta_s$  value from the inverse velocity plot, the  $\delta_{100}$  value from the velocity plot, and the  $c_v/d^2$  value from the  $t_{50}$  intercept; and the third, combined, response using the  $\delta_s$  from the conventional Taylor method, the  $\delta_{100}$  value from the velocity plot, and the average  $c_v/d^2$  value found above. A residual of zero indicates a fitted response which perfectly matches the observed response at the observed points. The fitted response using the combined parameters is also shown on the Taylor and Casagrande plots, Figs 1(a) and 1(b). Qualitatively, the fit is excellent until secondary compression effects become noticeable.

For each fitted response, the relative residual is calculated by dividing the actual residual by the absolute value of  $\delta_{100} - \delta_s$ . Using the relative residual helps to indicate whether the difference between the fitted and the observed values is large in comparison to the change in the observed values over time. Fig. 2(a) is a plot against timefactor of the relative residual for the three fitted responses. There is a systematic deviation at both the start and at the end of the test, indicating that some significant effects are not properly accounted for in the model produced by the fitting procedure (Draper & Smith, 1998).

The residual uses parameters derived from a model fitting procedure based on a selected subset of the observed response. At the start of the test, the conventional Taylor method produces smaller residuals than the velocity method, while towards the end of the test there is little difference between the two residuals. Towards the end of this test, the relative residual grows linearly with the logarithm of time. This area of the response corresponds to the later parts of the test where the effects of secondary compression become relatively important. Fitting a straight line to this part of the residual data, as indicated in Fig. 2(a), gives an intercept time-factor of  $T_{v,IT} \approx 0.96$ . This should not be interpreted as the time-factor at which secondary compression starts, which cannot be reliably determined without pore pressure measurements (Berre & Iversen, 1972; Robinson, 2003). Rather, this time-factor intercept on the relative residual plot indicates the point where the observed response starts to deviate systematically from the modelled response, so this is the point where the effect on the overall compression response of secondary compression becomes significant in comparison with the effect of primary consolidation. A more complete model incorporating secondary compression effects will be needed if this part of the soil's behaviour is to be

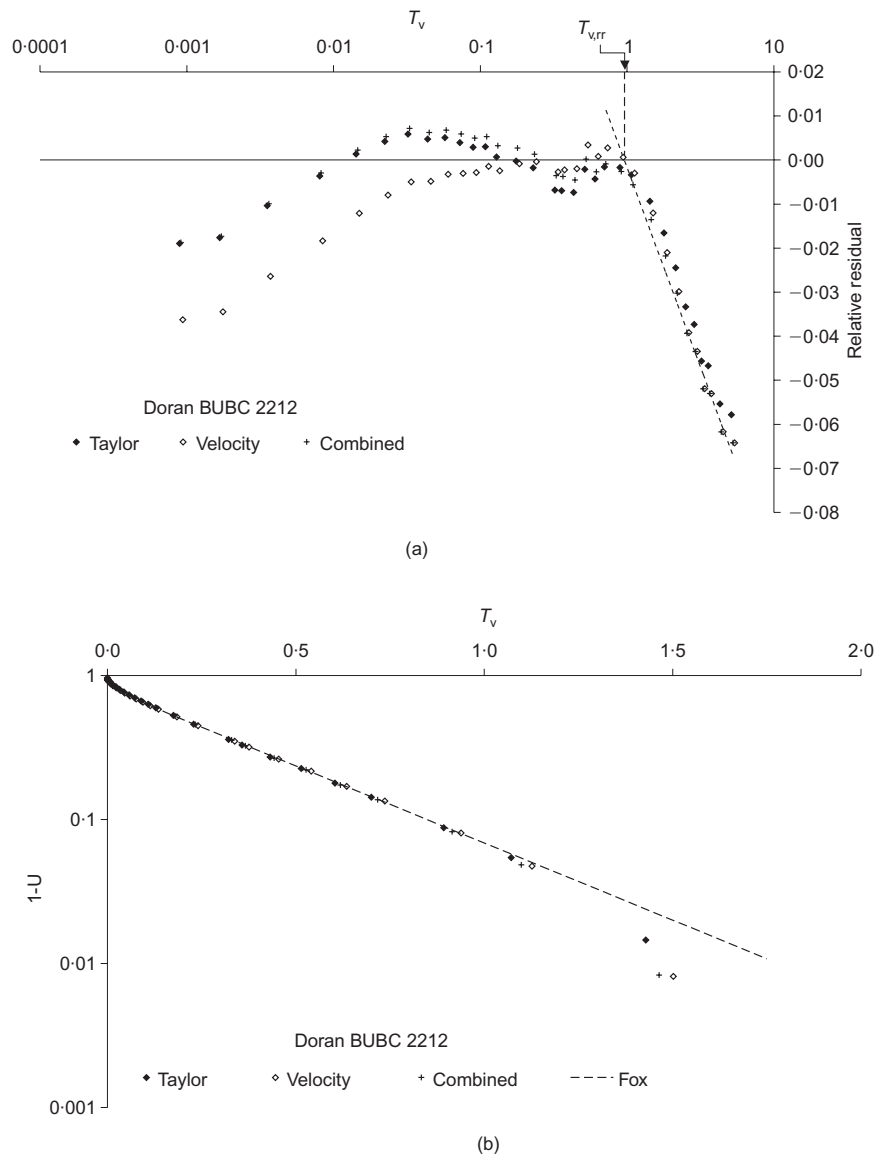


Fig. 2. Residual plot and Naylor and Doran plot for sample BUBC 2212: (a) residual plot; (b) Naylor and Doran plot

adequately represented. Consideration of this is beyond the scope of the current paper.

Naylor & Doran (1948) used a plot of  $1 - U$  against  $T_v$  as a means of determining the consolidation parameters. On this plot the theoretical primary consolidation response is essentially linear for  $U \geq 60\%$ , approximately. The linearity of the observed response on this plot is, however, very sensitive to the estimated value of  $\delta_{100}$  and the position of the observed response is moderately sensitive to the estimated values of  $\delta_s$  and  $c_v/d^2$ . While this makes the Naylor and Doran plot an awkward tool for determining  $\delta_s$ ,  $\delta_{100}$  and  $c_v/d^2$ , it does make it a useful tool for assessing the goodness of fit, particularly in relation to the estimate of  $\delta_{100}$ . Fig. 2(b) shows the Naylor and Doran plot for the three fitted responses for the sample Doran BUBC 2212, with the theoretical response calculated using Fox's approximate solution. Clearly, there is excellent agreement between the theoretical and the fitted responses, with little to choose between the three fitted responses. Note that the observed response cannot be plotted for displacement gauge readings beyond  $\delta_{100}$ , since  $1 - U$  would be negative, so the final point marked is just before the estimated end of primary consolidation.

#### DISCUSSION AND PROPOSED METHOD

The relative residual plot for sample Doran BUBC 2212 shows that the conventional Taylor approach produces more appropriate  $\delta_s$  value than the inverse velocity plot, as the former leads to lower residuals at the start of the test. Examination of the residuals plot for other consolidometer tests shows that this is generally the case. Moreover, estimation of  $\delta_s$  from the inverse velocity plot is a cumbersome procedure. There is good agreement on the Naylor and Doran plot between the fitted responses and the theoretical response, at least for  $T_v < 1$ . There is little difference between the three fitted responses generated, and examination of the Naylor and Doran plot for other conventional consolidometer tests shows similar agreement.

The current authors recommend the following method for the estimation of  $\delta_s$ ,  $\delta_{100}$  and  $c_v/d^2$  from a consolidometer test

- use  $\delta_s$  from the Taylor plot, using the conventional Taylor's method
- use  $\delta_{100}$  from the plot of velocity against displacement and



- (c) calculate all four estimates of  $c_v/d^2$  using the methods described above, and use the average value.

This method has been used to calculate the combined fitted response on Figs 1(a), 1(b), 2(a) and 2(b). For a simpler analysis, such as when undertaking the plotting and fitting by hand, the authors recommend estimating  $c_v/d^2$  using the  $t_{50}$  intercept method only. As a check on the estimates of  $\delta_s$  and  $\delta_{100}$  the authors recommend that  $\delta_s$  is calculated from the inverse velocity plot and  $\delta_{100}$  is calculated from the Taylor plot. The authors' analysis of other test results, illustrated by the example here, however, leads to the deduction that these should not be used to estimate final parameters.

The advantages of the proposed method for conventional consolidometer tests are twofold. First, the velocity plot leads to a reliable estimate of  $\delta_{100}$ , that can be compared with that calculated from  $t_{90}$  using the Taylor plot, as a qualitative check. Second, the full method generates four separate estimates of  $c_v/d^2$ , which can be compared as a qualitative check on reliability. However, quantitative measures of reliability, such as confidence limits, cannot be established from these four estimates in a robust way since they are not independent. As long as the four values are judged to be in reasonable agreement, it seems sensible to simply average the values. The subject of the reliability of the estimates of  $\delta_s$ ,  $\delta_{100}$  and  $c_v/d^2$  for a wider range of consolidation parameter estimation methods is one which the authors will address in a subsequent paper.

The method proposed involves the calculation of velocity from displacement measurements at time intervals, using a centred finite difference method, with the last 50% of primary consolidation being the area of interest. For realistic laboratory data the velocity estimates will be more noisy than the displacement measurements. If the time interval between readings becomes small, so that the change in displacement between readings is comparable to the noise in the displacement measurements, the velocity estimates will be dominated by the noise term. This can occur in modern laboratory tests with automated data collection. Some form of signal noise reduction would then be desirable. Data smoothing with a low pass digital filter, as adopted by McKinley (1993) for the processing of pore pressure data from self-weight sedimentation/consolidation tests in a geotechnical centrifuge, would be one approach. In the current authors' analysis of the results of conventional consolidometer tests, however, this has proved unnecessary.

## CONCLUSION

Manipulation of Fox's (1948) simple two-section approximate solution for Terzaghi's one-dimensional primary consolidation problem with an initially uniform excess pore pressure distribution indicates that a plot of velocity against displacement should be a straight line for a degree of consolidation in the range 50% to 100% approximately, and that the intercept on the velocity axis is  $\delta_{100}$ . The authors propose a velocity-displacement method in which  $\delta_s$  is estimated from a conventional Taylor plot,  $\delta_{100}$  is estimated from a plot of velocity against displacement, and four separate values for  $c_v/d^2$  are estimated from intersections and gradients on both plots. The method is implemented as a semi-automated Microsoft Excel spreadsheet, and its use is illustrated by application to laboratory test data for a conventional consolidometer test on a reconstituted Belfast Upper Boulder Clay. Experience with the method indicates that it is a robust means of estimating these important consolidation parameters.

Goodness of fit measures indicate that the method produces good agreement between fitted and measured displacements, up until the point where the impact of secondary compression on the overall response becomes significant. The method will be sensitive to noise in the displacement measurements, because it involves the calculation of velocity, but for the test data examined the sampling rate is sufficiently low for this to be unimportant. The proposed method provides a qualitative check on  $\delta_s$ ,  $\delta_{100}$  and  $c_v/d^2$ .

## ACKNOWLEDGEMENTS

The theoretical development presented here is based on an analysis originally undertaken by our former colleague Dr Gregg Doran, with whom the authors have had extensive and fruitful discussions concerning both this method and wider aspects of regression analysis for the determination of parameters from consolidometer data. Unfortunately, he died before the work reported here could be completed, so the authors were unable to draw on his expertise in the drafting of this paper.

## NOTATION

$c_v$	coefficient of vertical consolidation
$d$	length of drainage path
$t$	time since start of test
$t_{50}$	time at 50% of primary consolidation
$t_{90}$	time at 90% of primary consolidation
$t_{100}$	time at 100% of primary consolidation
$T_v$	timefactor
$T_{v,rr}$	timefactor intercept on relative residuals plot
$U$	degree of consolidation
$\delta$	displacement gauge reading
$\delta_s$	corrected zero point displacement gauge reading
$\delta_{50}$	displacement gauge reading at 50% of primary consolidation
$\delta_{100}$	displacement gauge at 100% of primary consolidation

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